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Instituto de Física de São Carlos

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Classical solutions of Yang-Mills theory

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Abstract

In this work, we investigate the $SU(2)$ Yang-Mills theory and derive two classes of classical solutions of physical interest, namely, monopoles and instantons. The monopole solutions arise when we consider the Higgs mechanism, by topological arguments and the homotopy group formalism, we conclude that the original symmetry group $SU(2)$ is spontaneously broken down to $U(1)$, in the process, the mass of gauge particles is generated and the connection between topological charge and magnetic charge is established for the monopole. We calculate the magnetic charge and the mass of the 't Hooft-Polyakov monopole. Then, we explore the Bogomolny bound and the Bogomolny-Prasad-Sommerfield monopole, which is an exact solution. We briefly discuss the Montonen-Olive duality. For instanton solutions, we consider a pure $SU(2)$ Yang-Mills theory on a Euclidean space-time, we show that the self-duality condition minimizes the Euclidean action, hence, we construct self-dual instanton and N -instanton solutions and calculate their action, obtaining that the action of N -instantons is n times the action of a single instanton.

Keywords: Yang-Mills theory. Classical solutions. Monopoles and instantons.

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1 Introduction

In 1954, Yang and Mills (1) proposed a field theory to study strong interactions, however, the gauge invariance of the theory required that the particles were massless. It was later found that mass could be generated through symmetry-breaking. The story of classical solutions began earlier, in 1931, when Dirac (2) started the quest for magnetic monopoles. Dirac showed that the existence of magnetic monopoles could explain the quantization of electric charge, however, he could not provide an estimation for the mass of the monopoles, and since they were not experimentally observed, his ideas were left aside. The theory of magnetic monopoles went through important development throughout the '70s, when 't Hooft (3) and Polyakov (4) showed that monopoles appear as a soliton solution in Yang-Mills theories with symmetry spontaneously broken by the Higgs mechanism (5), in their solution, the magnetic charge appears as a topological charge and its conservation is a consequence of the configuration of fields at spatial infinity, rather than of the dynamics. The quantization of magnetic charge for the 't Hooft-Polyakov monopole comes from topological arguments and yet they satisfy Dirac's quantization condition, 't Hooft and Polyakov were able to calculate a finite size and mass for their solution. Another important class of monopoles also developed in the same decade are the Bogomolny-Prasad-Sommerfield (BPS) monopoles (6), they appear when the Higgs potential vanishes in a way that the vacuum expectation value is different from zero, which maintains the broken symmetry. In the same decade, Belavin (7) showed that Yang-Mills theories have Euclidean solutions known as instantons; Jackiw, Rebbi (8), and 't Hooft (9) showed that instantons reveal a rich vacuum structure of quantum Yang-Mills theories and Polyakov suggested that instantons play a role in confinement (10). Those developments led Montonen and Olive (11) to conjecture, in the late '70s, that magnetic monopoles are gauge particles of the dual theory to which they appear as solitons, where the gauge coupling of the dual theory is the inverse of the usual gauge coupling.

Another chapter on monopole theory began in the early '90s when Seiberg and Witten (12) showed that the Montonen-Olive duality holds in a $SU(2)$ $N=2$ Super-Yang-Mills theory. In 1997, Maldacena (13) proposed the ground-breaking *AdS/CFT* correspondence, which consists of a duality between a kind of string theory oscillating in the bulk of an Anti-deSitter background compactified in a five-dimensional sphere through $AdS_5 \times S^5$ and a $\mathcal{N} = 4$ Super-Yang-Mills theory in the four-dimensional boundary, instanton solutions were found in this context (14). In

recent years, there are exciting developments in both theoretical and experimental frontiers in monopole research. The pioneer ideas of Montonen and Olive led to some of the most active areas of research in theoretical physics, known as the gauge/gravity duality, which consists of studying duality symmetries between string theory and quantum field theory. On the experimental side, the on-going MoEDAL (Monopole and Exotics Detector at the LHC) experiment has an unprecedented precision (15) and may lead to discoveries related to monopoles in the future.

2 $SU(2)$ Yang-Mills-Higgs solutions

The starting point is the field theory first proposed by Yang and Mills (1) in 1954, it is possible to construct a Yang-Mills theory for any compact Lie group G and a given Lie algebra \mathfrak{g} . The gauge invariance of pure Yang-Mills theories requires the gauge fields to be massless, however, it is possible to generate mass through the Higgs mechanism (5). From now on, we consider our gauge group to be $SU(2)$ whose algebra obeys:

$$[T_a, T_b] = i\epsilon_{abc}T_c, \quad \text{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}, \quad a, b, c = 1, 2, 3 \quad (1)$$

where ϵ_{abc} are the totally antisymmetric structure constants of the algebra and δ_{ab} is a Kronecker delta. The rich monopole phenomenology arises from the $SU(2)$ Yang-Mills theory together with the Higgs potential and the scalar fields in the adjoint representation, they will break the symmetry from $SU(2)$ to $U(1)$. We consider a model with the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} + (\mathcal{D}^\mu\phi_a)^\dagger\mathcal{D}_\mu\phi_a + V(\phi), \quad V(\phi) = \frac{\lambda}{4}(|\phi|^2 - a^2)^2. \quad (2)$$

where $a = 1, 2, 3$ are the $SU(2)$ group indices, the Higgs potential $V(\phi)$ is gauge invariant and renormalizable. It is necessary to introduce gauge fields A_μ^a in the minimal coupling of the fields ϕ_a to preserve local gauge invariance, therefore, the covariant derivative in terms of the group indices is defined as $(\mathcal{D}^\mu\phi)_a := \partial^\mu\phi_a - e\epsilon_{abc}A_b^\mu\phi_c$. The field tensor¹ is gauge invariant and is defined as $G_a^{\mu\nu} := \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - e\epsilon_{abc}A_b^\mu A_c^\nu$. The equations of motion of the system are:

$$(\mathcal{D}_\nu G^{\mu\nu})_a = -e\epsilon_{abc}\phi_b(\mathcal{D}^\mu\phi)_c \quad (\mathcal{D}^\mu\mathcal{D}_\mu\phi)_a = -\lambda\phi_a(\phi^2 - a^2) \quad (3)$$

those are the Yang-Mills equations and are a non-Abelian generalization of Maxwell's equations.

2.1 Homotopy groups and topological charges

The solutions of the theory (2) that may have some physical relevance must have finite energy. Therefore, in a given time, it is expected that the following conditions

$$V(\phi) = 0, \quad \mathcal{D}^\mu\phi_a = 0, \quad (4)$$

hold everywhere in space, except in a finite number of compact regions that will be called *monopoles*. Any region in space that satisfies the condition (4) will be in the *Higgs vacuum*.

¹ Notice that both the gauge fields and the field tensor belong to the algebra.

Since $V(\phi)$ is invariant under the action of group G , if ϕ_0 satisfies the condition (4) so does $D(g)\phi_0, \forall g \in G$, where D is a faithful representation. Then, we define the *vacuum manifold*:

$$\mathcal{M}_0 := \{\phi \mid V(\phi) = 0\}. \quad (5)$$

We say that two points ϕ_1 and ϕ_2 are in the same *orbit* if they can be related by $\phi_1 = D(g)\phi_2$. From now on, we include another hypothesis over \mathcal{M}_0 : it consists of a single orbit of the gauge group. Another way of expressing it is: G acts *transitively* on \mathcal{M}_0 , that is, $\phi_1, \phi_2 \in \mathcal{M}_0, \exists g_{12} \in G \mid \phi_1 = D(g_{12})\phi_2$. A very important concept is the *little group*. For a given $\phi \in \mathcal{M}_0$, we define:

$$H_\phi := \{h \in G \mid D(h)\phi = \phi\}. \quad (6)$$

The transitivity condition on the action of G implies that the structure of \mathcal{M}_0 is isomorphic to the right coset

$$\mathcal{M}_0 = G/H \quad (7)$$

where $H = H_{\phi_0}$ for a given $\phi_0 \in \mathcal{M}_0$. Since all Higgs vacua constitute only a single orbit of the gauge group, then all little groups are isomorphic. The little group has some very interesting physical properties, notably, its dimension gives the number of massless gauge particles and the eigenvalues of its generators determine possible values for different physical charges. If the system is in the vacuum, then the gauge symmetry left is inherited from the little group. The generators of H_ϕ are those that annihilate ϕ , and the only non-vanishing components of the gauge tensor field are those corresponding to H_ϕ . Thus, only the gauge fields H permeate the Higgs vacua. Going back to the concrete case of $SU(2)$, a given finite energy solution defines a map from a sphere with a large radius at spatial infinity to the vacuum manifold:

$$\phi : S_\infty^2 \rightarrow \mathcal{M}_0. \quad (8)$$

Those maps are characterized by an integer number called the *winding number*, this number is a topological invariant and has the interpretation of *topological charge*; two maps with the same index are said *homotopic* and can be continuously deformed into one another. The maps of the kind (8) can be classified into the homotopy group $\Pi_2(\mathcal{M}_0)$. Given the structure of the vacuum manifold, we have:

$$\Pi_2(\mathcal{M}_0) = \Pi_2(G/H) = \Pi_1(H), \quad (9)$$

where the last equality follows from the fact that G is simply-connected. If H is not simply-connected, then $\Pi_1(H)$ has a non-trivial structure and the theory may have monopole solutions.

Furthermore, in our case, where $G = SU(2)$ and $H = U(1)$, we have:

$$\Pi_2[SU(2)/U(1)] = \Pi_1[U(1)] = \mathbb{Z} \quad (10)$$

and thus our model (2) can hold monopole solutions with integer topological charge.

2.2 The Higgs mechanism and the masses of gauge particles

Given the topological considerations about the symmetry groups in the previous section, we conclude that the original symmetry group $G = SU(2)$ is spontaneously broken down to $H = U(1)$ by ϕ . Nevertheless, we end up with a $U(1)$ gauge theory which can be identified with the traditional electromagnetism. In view of the generators of $SU(2)$, we see that the generator of $U(1)$ is given by $\phi \cdot T/a$, which is proportional to the electric charge. Furthermore, the vacuum expectation value of the scalar field is given by $\phi_0 = a$, if we consider small fluctuations around the vacua, we may write $\phi = \phi_0 + \eta$. Substituting that into the Lagrangian yields a mass term for the gauge particles coming from the coupling of the scalar fields to the gauge fields:

$$(\mathcal{D}_\mu \phi_0)^\dagger \mathcal{D}^\mu \phi_0 = e^2 \phi_0^\dagger D(A_\mu) D(A^\mu) \phi_0 = \frac{1}{2\hbar^2} M_{ab}^2 A_\mu^a A^{b\mu},$$

we recognize the mass in the usual way, as the square root of the coefficients of the quadratic terms in the bosonic fields (and the potential) divided by $2\hbar^2$:

$$M_{ab}^2 = e^2 \hbar^2 \phi_0^\dagger D(\{T_a, T_b\}) \phi_0, \quad (11)$$

for the generators of the little group, it follows from (6) that $D(T)\phi_0 = 0$, yielding null mass for the corresponding gauge particle², while the remaining gauge bosons obtain a mass $M = ae\hbar$.

2.3 The connection between topological and magnetic charges

In this section we want to make the connection between topological charge and magnetic charge. A very general configuration of the fields that satisfy the condition (4) is given by (16):

$$\vec{A}_\mu = \frac{1}{a^2 e} \vec{\phi} \times \partial^\mu \vec{\phi} + \frac{1}{a} \vec{\phi} B^\mu, \quad \vec{\phi}^2 = a^2 \quad (12)$$

where B^μ is an arbitrary four-vector and the term $\frac{1}{a} \vec{\phi} B^\mu$ belongs to the algebra. Since the gauge group and the adjoint representation of the scalar field are three vectors with respect to the

² As we have seen, the little group has $U(1)$ symmetry, therefore, the associated gauge particle is the photon, which indeed has null mass.

gauge group indices, we use the usual vector notation. The field strength tensor becomes:

$$\vec{G}^{\mu\nu} = \frac{1}{a}\vec{\phi}F^{\mu\nu}, \quad F^{\mu\nu} = \frac{1}{a^3e}\vec{\phi} \cdot (\partial^\mu\vec{\phi} \times \partial^\nu\vec{\phi}) + \partial^\mu A^\nu - \partial^\nu A^\mu \quad (13)$$

given the tensor $F^{\mu\nu}$ and the configuration of the fields (12), we recover the equations of Maxwell:

$$\partial_\nu F^{\mu\nu} = 0, \quad \partial^\nu \tilde{F}^{\mu\nu} = 0, \quad (14)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ denotes the Hodge dual. We have the following conclusion: in the Higgs vacuum, the only non-vanishing component of the field tensor is the one related to $U(1)$, the group of rotations about the axis defined by $\vec{\phi}$, which is the little group of the vacuum. On top of that, this component satisfies Maxwell's equations. The magnetic flux g_Σ is only non-zero if the closed surface Σ surrounds a region in which the vacuum conditions (4) are not satisfied, that is, a monopole region. It follows:

$$g_\Sigma = \int_\Sigma \vec{B} \cdot d\vec{s} = \frac{1}{2}\epsilon_{ijk} \int_\Sigma F^{jk} ds^i, \quad (15)$$

the contribution of the partial derivatives in (13) vanishes by the Stokes' Theorem. We obtain:

$$g_\Sigma = \frac{1}{2ea^3} \int_\Sigma \epsilon_{ijk} \vec{\phi} \cdot (\partial^j \vec{\phi} \times \partial^k \vec{\phi}) ds^i. \quad (16)$$

Consider small homotopic variations on ϕ , that is, $\vec{\phi}' = \vec{\phi} + \delta\vec{\phi}$, the variation of the integrand on the previous equation yields

$$\delta[\vec{\phi} \cdot (\partial^j \vec{\phi} \times \partial^k \vec{\phi})] = \partial^j [\vec{\phi} \cdot (\delta\vec{\phi} \times \partial^k \vec{\phi})] - \partial^k [\vec{\phi} \cdot (\delta\vec{\phi} \times \partial^j \vec{\phi})],$$

however, such contribution vanishes due to Stokes' theorem. We conclude that under homotopic deformations, the magnetic flux remains unaltered, some examples of homotopic deformations include time evolution and continuous gauge transformations, this means that g_Σ is time-independent and gauge invariant. Note that g_Σ depends on the homotopy classes of the mapping $\Sigma \rightarrow \mathcal{M}_0$. However, as discussed in Section 2.1, we have $\Sigma = \mathcal{M}_0 = S^2$, where equality denotes isomorphism, and thus the homotopy group is $\Pi_2(S^2) = \mathbb{Z}$, where the integer labeling the homotopy group gives the number of times one sphere wraps around the other given the mapping $\Sigma \rightarrow \mathcal{M}_0$. We may rewrite g_Σ in the following way:

$$g_\Sigma = \frac{4\pi}{e}N, \quad N := \frac{1}{8\pi a^3} \int_\Sigma \epsilon_{ijk} \vec{\phi} \cdot (\partial^j \vec{\phi} \times \partial^k \vec{\phi}) ds^i \quad (17)$$

which means $\vec{\phi}$ covers the sphere \mathcal{M}_0 in a total of N times as \hat{r} covers Σ once. The index N is the winding number we referred to previously. The magnetic charge depends only on the homotopy

class of the map $\phi : S^2 \rightarrow \mathcal{M}_0$. Considering the smallest electric charge of the theory as $q = \frac{1}{2}e\hbar$, we recover Dirac's (2) quantization condition $qg = 2\pi N\hbar$, which concludes the relation between topological charges, that come from topology considerations and homotopy theory, and magnetic charges, that come from the Higgs vacuum structure and Maxwell's equations.

2.4 The 't Hooft-Polyakov solution

A remarkable solution of the model (2), was done independently by 't Hooft (3) and Polyakov (4) in 1974. The starting point is the ansatz:

$$\phi_a = \frac{r^a}{er^2}H(aer), \quad A_a^i = -\epsilon_{aij}\frac{r^j}{er^2}[1 - K(aer)], \quad A_a^0 = 0. \quad (18)$$

where $a = 1, 2, 3$ are the group indices and $i, j = 1, 2, 3$ are the spatial indices, $H(aer)$ and $K(aer)$ are smooth functions. The asymptotic behavior (4) for $V(\phi) \rightarrow 0$ implies $\phi_a(\vec{r}) = \lim_{r \rightarrow \infty} \phi_a(r\hat{r}) = a\hat{r}^a$ and thus ϕ maps each point in the sphere S^2 to the corresponding point in the manifold³ $\mathcal{M}_0 = S^2$. Those maps cannot be continuously deformed into a constant map that satisfies (4), accordingly, the monopole is topologically stable and cannot decay into the vacuum state. Defining $\xi := aer$, the equations of motion for the ansatz become:

$$\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2 - 1) \quad \xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \quad (19)$$

with the boundary conditions for a finite energy solution:

$$K(\xi) \rightarrow 1, \quad H(\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow 0; \quad (20)$$

$$K(\xi) \rightarrow 0, \quad H(\xi) \rightarrow \xi \quad \text{as } \xi \rightarrow \infty. \quad (21)$$

The energy of the system is given by

$$\begin{aligned} E &= \int d^3r \mathcal{H} = \int d^3r (p_i \dot{q}_i - \mathcal{L}) = - \int d^3r \mathcal{L} \\ &= \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2} \left[\xi^2 \left(\frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left(\xi \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2} (K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \right], \end{aligned}$$

interpreting the total energy as the mass, we have $M = \frac{4\pi a}{e} f(\lambda/e^2)$, where $f(\lambda/e^2)$ is a slowly-varying function of the coupling parameters of the theory. By using the ansatz (18) in the

³ That can easily be seen through the defining equation $\phi_1^2 + \phi_2^2 + \phi_3^2 = a^2$.

covariant derivative of the scalar field and in the non-Abelian magnetic field, we obtain

$$\mathcal{D}_i \phi^a = \delta_{ai} \frac{KH}{er^2} + \frac{r^a r^i}{er^4} \left(\xi \frac{dH}{d\xi} - H - KH \right) \xrightarrow{\xi \gg 1} 0, \quad (22)$$

$$B_i^a = \frac{1}{2} \epsilon_{ijk} G_{jk}^a = \frac{r_i r^a}{er^4} \left(1 - K^2 + \xi \frac{dK}{d\xi} \right) - \delta_{ai} \frac{\xi}{er^4} \frac{dK}{d\xi} \xrightarrow{\xi \gg 1} \frac{r_i r^a}{er^4}. \quad (23)$$

where $\xi \propto r$ and the conditions $\xi \gg 1$ and $r \gg 1$ are equivalent. The condition $\mathcal{D}_i \phi^a \rightarrow 0$ is indeed satisfied, as required for finite energy. We have an asymptotically Coulomb-like magnetic field $\vec{B} \sim (1/er^2)\hat{r}$. That suggests a comparison with Dirac's monopole, whose magnetic field is given by $\vec{B} = [g/(4\pi r^2)]\hat{r}$, however, the 't Hooft-Polyakov monopole has smooth internal structure and no singularities (18). When comparing the charges, we see that $g = 4\pi/e$, which agrees with the $N = 1$ sector for the equation (17), as expected from our topological arguments. On top of that, it also agrees with Dirac's quantization condition $qg = 2\pi\hbar$ where $q = \frac{1}{2}e\hbar$. In this sense, the 't Hooft-Polyakov carries one unit of magnetic charge.

2.5 Bogomolny bound and the mass of the 't Hooft-Polyakov monopole

An important feature of monopoles with a smooth internal structure is the fact that their mass can be calculated. Following our discussion on Section 2.3, we have the definition of the electromagnetic tensor (13). For any finite energy solution, the magnetic charge associated with the $U(1)$ component is

$$g = \int \vec{B} \cdot d\vec{S} = \frac{1}{a} \int B_a^k \phi_a dS^k = \frac{1}{a} \int B_a^k (\mathcal{D}^k \phi)_a d^3r \quad (24)$$

where the surface integral is over the sphere at spatial infinity. B_a^k is given by $G_a^{ij} = -\epsilon_{ijk} B_a^k$. Similarly, from the equations of motion (3), we identify the electric charge

$$q = \int \vec{E} \cdot d\vec{S} = \frac{1}{a} \int E_a^k (\mathcal{D}^k \phi)_a d^3r, \quad (25)$$

where $G_a^{0k} = -E_a^k$. Consider the center-of-mass frame of the monopole. Its mass is given by

$$M = \int d^3r \frac{1}{2} \left[(E_a^k)^2 + (B_a^k)^2 + (\mathcal{D}^0 \vec{\phi})^2 + (\mathcal{D}^i \vec{\phi})^2 + 2V(\phi) \right] \quad (26)$$

$$\geq \int d^3r \frac{1}{2} \left[(E_a^k)^2 + (B_a^k)^2 + (\mathcal{D}^i \vec{\phi})^2 \right] \quad (27)$$

considering the equations for the electric charge (25) and for the magnetic charge (24) we may rewrite the inequality (27) in the following way

$$M \geq \frac{1}{2} \int d^3r [E_a^k - (\mathcal{D}^k \phi)_a \sin \theta]^2 + \frac{1}{2} \int d^3r [B_a^k - (\mathcal{D}^k \phi)_a \cos \theta]^2 + a(q \sin \theta + g \cos \theta) \quad (28)$$

$$\geq a(q \sin \theta + g \cos \theta) \quad (29)$$

for any real angle θ . The sharpest limit happens when⁴

$$M \geq a\sqrt{q^2 + g^2} \quad (30)$$

this inequality is called the Bogomolny bound (17). As an example, we may consider the 't Hooft-Polyakov monopole discussed on Section 2.4. We saw that this solution has magnetic charge $|g| = 4\pi/e$ and null electric charge, hence, its mass is $M_g \geq 4\pi a/e$. If we compare the calculated mass of the monopole to the mass of gauge particles $M = ae\hbar = aq$, we have:

$$M_g \geq \frac{4\pi\hbar}{q^2} M \simeq 137M, \quad (31)$$

the mass of the 't Hooft-Polyakov is more than two orders of magnitude larger than the mass of the gauge bosons.

2.6 The Bogomolny-Prasad-Sommerfield solution

Now we want to consider solutions where the Bogomolny bound (30) is saturated. This solution must be a finite-energy one, thus, the following conditions must hold throughout space:

$$\mathcal{D}^0 \vec{\phi} = 0, \quad E_a^k = 0, \quad B_a^k = \pm(\mathcal{D}^k \phi)_a \quad \text{and} \quad V(\phi) = 0. \quad (32)$$

the condition on the Higgs potential can only be satisfied if the coupling λ goes to zero. However, such condition can be understood as a limit, were we take $|\vec{\phi}| \rightarrow a$ as $r \rightarrow \infty$ and $\lambda \rightarrow 0$, this limit guarantees that the charges remain properly quantised and it reduces, together with conditions (32), the equations of motion (3) to

$$\mathcal{D}_\nu G^{\mu\nu} = e\vec{\phi} \times \mathcal{D}^\mu \vec{\phi} \quad \text{and} \quad \mathcal{D}_\mu \mathcal{D}^\mu \vec{\phi} = 0. \quad (33)$$

Inserting the ansatz (18) on the new equations of motion (33) yields:

$$\xi \frac{dK}{d\xi} = -KH \quad \text{and} \quad \xi \frac{dH}{d\xi} = H - (K^2 - 1) \quad (34)$$

where $\xi = aer$. An exact solution of those two equations was found by Prasad and Sommerfield (6), it is given by

$$H(\xi) = \xi \coth \xi - 1 \quad \text{and} \quad K(\xi) = \frac{\xi}{\sinh \xi} \quad (35)$$

⁴ The maximum value of $q \sin \theta + g \cos \theta$ happens when $\theta = \varphi$ such that $\tan \varphi = q/g$. This is equivalent to $\sqrt{q^2 + g^2} = q \sin \varphi + g \cos \varphi$.

this solution corresponds to a monopole with magnetic charge $g = 4\pi/e$ and mass $M = ag$, the greatest feature of this solution is that it is exact. As $\xi \rightarrow \infty$, the function H approaches its asymptotic form as $H = 1 + \xi + \mathcal{O}(\exp(-\xi))$, this means that the Higgs approaches the vacuum at large distances in a much slower fashion than in the 't Hooft-Polyakov solution, that is, although there is a spontaneous breakdown of symmetry generating mass to the gauge fields, the scalar fields themselves are massless.

The BPS monopoles and the Bogomolny mass formula (30) play a fundamental role in duality conjectures. We have seen that gauge theories with broken symmetry by the Higgs have magnetically neutral particles with mass $M = aq = ae\hbar$, those are interpreted as quantum fluctuations of the fields; we also have magnetically charged particles with mass $M = ag$ that appear as classical solutions of the equations of motion. However, despite its different nature, both kinds of particles satisfy the Bogomolny mass formula (equality of eq. 30). A completely analogous duality happens in the sine-Gordon model and the massive Thirring model, where solitons of the sine-Gordon model are created by a quantum field obeying the equations of motion of the Thirring model (18). Those considerations motivated Olive and Montonen (11) to conjecture that there exists a quantum field operator that creates monopole solitons (like the BPS monopoles) and couples to a magnetic gauge group with strength inversely related to the original electric gauge coupling. In this case, solitonic monopoles would appear as gauge particles in the dual theory and vice-versa. In principle, this conjecture is sound because it is compatible with the tree-level interactions of the particles. Indeed, the massless Higgs particles mediate an attractive force that exactly cancels the repulsion between gauge particles of equal charge and doubles the attractive force between gauge particles of opposite charges. That is exactly what is observed in the interaction of the BPS monopoles among themselves.

The Montonen-Olive duality found success once supersymmetry was included, as shown by Seiberg and Witten (12); however, whether or not supersymmetry is an actual symmetry of nature remains open. The pioneer ideas of Olive and Montonen about duality nurtured further duality conjectures, which remains a very active topic of research.

3 Euclidean $SU(2)$ Yang-Mills solutions

In this section, we construct instantons and solutions which have led to new insights into a wide range of phenomena, such as revealing the structure of the Yang-Mills vacuum (8), (9), confinement (10) and string theory applications. Our theory consists on an Euclidean space-time \mathbb{E}^4 with coordinates $x = (x_1, x_2, x_3, x_4)$ and a pure Yang-Mills theory with gauge group $G = SU(2)$. For simplicity of notation, we take

$$A_\mu(x) = \sum_a \frac{e}{2i} A_\mu^a(x) T^a, \quad G_{\mu\nu} = \sum_a \frac{e}{2i} G_{\mu\nu}^a T^a \quad (36)$$

where A_μ^a are the gauge fields, $G_{\mu\nu}^a$ are the fields tensor, and T^a are the generators of the $\mathfrak{su}(2)$ algebra. The Euclidean action is given⁵ by

$$S = -\frac{1}{2e^2} \int d^4x \text{Tr}(G_{\mu\nu} G_{\mu\nu}) \quad (37)$$

and the resulting Yang-Mills equation becomes

$$\mathcal{D}_\mu G_{\mu\nu} = \partial_\mu G_{\mu\nu} + [A_\mu, G_{\mu\nu}] = 0. \quad (38)$$

As we will see, instantons will be finite-action solutions of the equation (38). A similar approach from the one used in the study of monopoles will be made for Euclidean solutions, we start by analyzing the boundary conditions that must be satisfied by any field configuration with finite-action, based on those conditions, we classify the solutions through homotopy theory.

As a first step, let us consider configurations of null action, that is, $S = 0$. This implies that $G_{\mu\nu} = 0$, which is a gauge-invariant condition because $G_{\mu\nu} \rightarrow \omega G_{\mu\nu} \omega^{-1}$ for a given gauge transformation ω , notice that such condition is satisfied by any gauge-transformed field from $A_\mu = 0$. Those fields are called pure gauges and are given by $A_\mu(x) = \omega \partial_\mu \omega^{-1}$, where ω is an element of a matrix representation of the $\mathfrak{su}(2)$ algebra. Going back to finite-action configurations, it is clear from (37) that $G_{\mu\nu}$ must go to zero in the boundary of \mathbb{E}^4 , that is, if we have S^3 as $r \rightarrow \infty$ where $|r| = |x| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$, then $G_{\mu\nu} \rightarrow 0$ must be faster than $\propto 1/x^2$. From the fact that $G_{\mu\nu}$ goes to zero in the boundary S^3 , we have the pure gauge condition: $A_\mu(x) \rightarrow \omega \partial_\mu \omega^{-1}$ as $x \rightarrow \infty$. By the definition of $SU(2)$, the representation matrices ω constitute a set of unitary 2×2 matrices with unity determinant. They can be written as:

$$\omega = a_\mu s_\mu, \quad a_\mu a_\mu = 1 \quad (39)$$

⁵ Since we are working in a Euclidean space, covariant and contravariant indices no longer differ. Einstein's sum convention is still implied whenever an index is repeated twice unless explicitly said otherwise. The Euclidean action is defined as the analytic continuation of the Minkowski action.

where s_4 is the identity 2×2 matrix, and $s_k = i\sigma_k$, where $k = 1, 2, 3$ and σ_k denote the Pauli matrices. a_μ are any real numbers satisfying the constrain (39). The group space is therefore a three-dimensional surface of a four-dimensional unitary sphere S_{int}^3 , thus, the functions ω are maps $S^3 \rightarrow S_{\text{int}}^3$, such mappings correspond to the homotopy group $\Pi_3(S^3) = \mathbb{Z}$; mappings of this type can be classified by discrete homotopy classes, labeled by the *Pontryagin index*. Following the discussion from Section 2.1, a field $A_\mu(x)$ on the four-space that belongs to a N -sector cannot be continuously deformed into another sector without violating the finiteness of the action, giving the solution topological stability. The Pontryagin index may be written explicitly in terms of the fields tensor:

$$N = -\frac{1}{16\pi^2} \int d^4x \text{Tr}(G_{\mu\nu} \tilde{G}_{\mu\nu}) \quad (40)$$

where $\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ denotes the Hodge dual.

3.1 Instanton solutions

Instantons are classical, localized, and finite-action solutions of the Euclidean Yang-Mills equation (38). We begin by considering the identity:

$$S = -\frac{1}{2e^2} \int d^4x \text{Tr}(G_{\mu\nu}G^{\mu\nu}) = -\frac{1}{4e^2} \int d^4x \text{Tr}(G_{\mu\nu} \mp \tilde{G}_{\mu\nu})^2 \pm 2\text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}) \quad (41)$$

note that $\text{Tr}(G_{\mu\nu}G_{\mu\nu}) = \text{Tr}(\tilde{G}_{\mu\nu}\tilde{G}_{\mu\nu})$. From equation (41) we see that the minimum of the action is obtained in a given sector when

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha\beta} = \pm G_{\mu\nu}. \quad (42)$$

Therefore, the self-duality (or anti-self-duality) condition (42) extremizes the action and thus gives us solutions of equation (38). Now, we look for solutions with self-duality following a method that was developed by 't Hooft (19), Corrigan and Fairlie (20), and Belavin et al (7). Consider the following ansatz for the gauge field:

$$A_\mu(x) = i\bar{\Sigma}_{\mu\nu}\partial_\nu \ln \phi(x) \quad (43)$$

where $\phi(x)$ is a function to be determined and $\bar{\Sigma}_{\mu\nu}$ are the 't Hooft symbols:

$$\bar{\Sigma}_{\mu\nu} = \bar{\eta}_{i\mu\nu} \frac{\sigma^i}{2}, \quad \text{where} \quad \bar{\eta}_{i\mu\nu} = -\bar{\eta}_{i\nu\mu} = \begin{cases} \epsilon_{i\mu\nu} & \text{for } \mu, \nu = 1, 2, 3 \\ -\delta_{i\mu} & \text{for } \nu = 4 \end{cases}$$

notice that $\bar{\Sigma}_{\mu\nu}$ is antisymmetric and antidual in the indices. It also has the following properties:

$$\begin{aligned} [\bar{\Sigma}_{\mu\sigma}, \bar{\Sigma}_{\nu\rho}] &= i(\delta_{\mu\nu}\bar{\Sigma}_{\sigma\rho} + \delta_{\rho\sigma}\bar{\Sigma}_{\mu\nu} - \delta_{\mu\rho}\bar{\Sigma}_{\sigma\nu} - \delta_{\nu\rho}\bar{\Sigma}_{\mu\sigma}) \\ \epsilon_{\mu\nu\alpha\beta}\bar{\Sigma}_{\beta\sigma} &= (\delta_{\mu\sigma}\bar{\Sigma}_{\nu\alpha} + \delta_{\nu\rho}\bar{\Sigma}_{\alpha\mu} + \delta_{\alpha\sigma}\bar{\Sigma}_{\mu\nu}), \quad \epsilon_{\mu\nu\alpha\beta}\bar{\Sigma}_{\alpha\beta} = -2\bar{\Sigma}_{\mu\nu} \end{aligned}$$

given the ansatz (43) for A_μ and the properties mentioned above, the field tensor is written as

$$G_{\mu\nu} = i\bar{\Sigma}_{\nu\sigma}(\partial_\mu\partial_\sigma\ln\phi - \partial_\mu\ln\phi\partial_\sigma\ln\phi) - i\bar{\Sigma}_{\mu\sigma}(\partial_\nu\partial_\sigma\ln\phi - \partial_\nu\ln\phi\partial_\sigma\ln\phi) - i\bar{\Sigma}_{\mu\nu}(\partial_\sigma\ln\phi)^2 \quad (44)$$

taking its dual, it follows

$$\begin{aligned} \tilde{G}_{\mu\nu} &= i\epsilon_{\mu\nu\alpha\beta}\left[\bar{\Sigma}_{\beta\sigma}(\partial_\alpha\partial_\sigma\ln\phi - \partial_\alpha\ln\phi\partial_\sigma\ln\phi) - \frac{1}{2}\bar{\Sigma}_{\alpha\beta}(\partial_\sigma\ln\phi)^2\right] \\ &= i\bar{\Sigma}_{\nu\alpha}(\partial_\alpha\partial_\mu\ln\phi - \partial_\alpha\ln\phi\partial_\mu\ln\phi) - i\bar{\Sigma}_{\mu\alpha}(\partial_\alpha\partial_\nu\ln\phi - \partial_\alpha\ln\phi\partial_\nu\ln\phi) + i\bar{\Sigma}_{\mu\nu}\partial_\sigma\partial_\sigma\ln\phi \quad (45) \end{aligned}$$

using the condition of self-duality yields from equations (44) and (45):

$$\partial_\mu\partial_\sigma\ln\phi - \partial_\mu\ln\phi\partial_\sigma\ln\phi = \partial_\sigma\partial_\mu\ln\phi - \partial_\sigma\ln\phi\partial_\mu\ln\phi \quad (46)$$

$$\partial_\sigma\partial_\sigma(\ln\phi) + (\partial_\sigma\ln\phi)^2 = \square\phi/\phi = 0 \quad (47)$$

note that equation (46) is trivially satisfied, and for the Euclidean metric $\square\phi = \partial_\sigma\partial_\sigma\phi$. We are looking for solutions of equation (47) to substitute in the ansatz (43). When ϕ is non-singular at the origin, it reduces to $\square\phi = 0$, which yields constant solutions for ϕ and null solutions for A_μ , those belong to the $N = 0$ sector and have trivial topological charge. Hence, we are interested on the solutions where ϕ is singular at the origin, in this case, we have interesting and non-zero solutions for the gauge fields. Let $y_\mu = x_\mu - a_\mu$ and $y^2 = y_\mu y_\mu$, consider the following form for $\phi(x)$:

$$\phi(x) = 1 + \frac{\lambda^2}{y^2} \quad (48)$$

we will show that it is an instanton solution that belongs to the $N = 1$ sector. The four-vector a_μ localizes the instanton and λ represents the size of the instanton in terms of the action density. An important property of pure Yang-Mills theories is their scale invariance. According to the ansatz (43), this yields the following gauge field

$$A_\mu(x) = -2i\lambda^2\bar{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2(y^2 + \lambda^2)}, \quad (49)$$

the gauge field is indeed singular at $y = 0$. However, such singularity may be removed by a corresponding singular gauge transformation. Given that both the self-duality condition and

the Yang-Mills equation are covariant, the resulting function will still be a solution of (38), minimizing the action. The required gauge function follow the form (39) and can be written as

$$\omega(x) = \frac{x_4 + ix_j \sigma_j}{|x|} = \hat{x}_\mu s_\mu \quad (50)$$

this is due to the pure gauge form:

$$[\omega(y)]^{-1} \frac{\partial}{\partial y_\mu} [\omega(y)] = -2i \bar{\Sigma}_{\mu\nu} \left(\frac{y_\nu}{y^2} \right) \quad (51)$$

hence, the gauge field (49) can be rewritten as

$$A_\mu(x) = [\omega(y)]^{-1} \partial_\mu [\omega(y)] \frac{\lambda^2}{y^2 + \lambda^2}. \quad (52)$$

Under a gauge transformation, the gauge fields behave as

$$\begin{aligned} A_\mu(x) \rightarrow A'_\mu(x) &= [\omega(y)] A_\mu [\omega(y)]^{-1} + [\omega(y)] \partial_\mu [\omega(y)]^{-1} \\ &= (\partial_\mu \omega) \omega^{-1} \left(\frac{\lambda^2}{y^2 + \lambda^2} - 1 \right) = -(\partial_\mu \omega) \omega^{-1} \frac{y^2}{y^2 + \lambda^2} = \omega (\partial_\mu \omega^{-1}) \frac{y^2}{y^2 + \lambda^2}. \end{aligned} \quad (53)$$

Using the self-duality for $\bar{\Sigma}_{\mu\nu}$, we have

$$\Sigma_{\mu\nu} = \eta_{i\mu\nu} \frac{\sigma^i}{2}, \quad \text{where} \quad \eta_{i\mu\nu} = -\eta_{i\nu\mu} = \begin{cases} \epsilon_{i\mu\nu} & \text{for } \mu, \nu = 1, 2, 3 \\ +\delta_{i\mu} & \text{for } \nu = 4 \end{cases}$$

notice that $\Sigma_{\mu\nu}$ also satisfies equation (51). The gauge transformed solution (53) can be written, in analogy to (49), as

$$A'_\mu(x) = -2i \Sigma_{\mu\nu} \frac{y_\nu}{y^2 + \lambda^2} = -2i \Sigma_{\mu\nu} \frac{(x-a)_\nu}{|(x-a)|^2 + \lambda^2}. \quad (54)$$

Although the solution we obtained is singular (49), a gauge transformation by a singular function (50) yields a non-singular solution. The field tensor related to $A'_\mu(x)$ is

$$G'_{\mu\nu} = 4i \Sigma_{\mu\nu} \frac{\lambda^2}{|x-a|^2 + \lambda^2}, \quad (55)$$

since $\Sigma_{\mu\nu}$ is self-dual so is $G'_{\mu\nu}$. Therefore, the transformed gauge field (54) is indeed a solution of the Yang-Mills equation that minimizes the action. Note that as $x \rightarrow \infty$, the solution indeed behaves as a pure gauge (51), that is, $A'_\mu(x) \rightarrow -2i \Sigma_{\mu\nu} = \omega \partial_\mu \omega^{-1}$, which is consistent with the finiteness of the action. In fact, the action of this solution can be calculated:

$$S = -\frac{1}{2e^2} \int d^4x \text{Tr}(G'_{\mu\nu} G'_{\mu\nu}) = \frac{48\lambda^4}{e^2} \int \frac{d^4y}{(y^2 + \lambda^2)} = \frac{8\pi^2}{e^2}, \quad (56)$$

because $G_{\mu\nu} = \tilde{G}_{\mu\nu}$ for this solution, the action has the form $S = 8\pi^2 N/e^2$, which confirms that our solution belongs to the $N = 1$ sector and has topological charge of one unit.

3.2 N -Instanton solutions

Now we look for multi-instanton solutions, our starting point is a generalization for N -instantons of the form (48):

$$\phi(x) = 1 + \sum_{i=1}^N \frac{\lambda_i^2}{|x_\mu - a_\mu|^2}. \quad (57)$$

The ansatz (43) yields the corresponding gauge fields:

$$A_\mu(x) = i\bar{\Sigma}_{\mu\nu}\partial_\nu \left[\ln \left(1 + \sum_{i=1}^N \frac{\lambda_i^2}{|y_i|^2} \right) \right] = -2i\bar{\Sigma}_{\mu\nu} \left(\sum_{i=1}^N \frac{\lambda_i y_{i\nu}}{|y_i|^4} \right) / \left[1 + \sum_{j=1}^N \frac{\lambda_j^2}{|y_j|^2} \right] \quad (58)$$

where $(y_i)_\mu = (x - a_i)_\mu$ with $i, j = 1, 2, \dots, N$. The pure gauge identity (51) still holds, which allows to rewrite the gauge field (58) as

$$A_\mu(x) = \sum_{i=1}^N \omega^{-1}(y_i) \partial_\mu \omega(y_i) \left[\frac{\lambda_i^2}{(x - a_i)^2} \right] / \left(1 + \sum_{j=1}^N \frac{\lambda_j^2}{(x - a_j)^2} \right) \quad (59)$$

the function $\omega(y_i)$ is the same as in the one instanton solution (50), built from the $SU(2)$ structure. The gauge field (58) is indeed a solution of the Yang-Mills equation because it satisfies the self-duality condition, however, similarly to the $N = 1$ solution, it is singular. In fact, it has N singularities at the points $x = a_i$, where $i = 1, \dots, N$. Near one of its singularities, the field behave as⁶

$$A_\mu(x) \rightarrow \omega^{-1}(x - a_i) \partial_\mu [\omega(x - a_i)], \quad (60)$$

that is, the N -instanton solution becomes a pure gauge when $x \rightarrow a_i$. As we did for the $N = 1$ solution, we may remove the singularities of the gauge field by a gauge transformation, this already hints at the solution as a collection of N single instantons. By the pure gauge condition (60), we know that the gauge function must reduce to $\omega(x - a_i)$ as x approaches a_i . Giambiagi and Rothe (21) proposed such function that belongs to a representation of $SU(2)$:

$$\omega_N(x) = \frac{z_4 + i\vec{z} \cdot \vec{\sigma}}{\sqrt{z^2}} \quad (61)$$

where $\vec{\sigma}$ is a vector of the Pauli matrices and z_μ is defined as

$$z_\mu(x) := \sum_{i=1}^N \alpha_i \begin{pmatrix} y_{i\mu} \\ y_i^2 \end{pmatrix}, \quad y_i := x_i - a_i, \quad (62)$$

⁶ Notice that $\partial_\mu = \frac{\partial}{\partial x_\mu}$ can be written as $\frac{\partial}{\partial(x - a_i)_\mu}$ since a_i is constant.

where α_i are real constants. As $y \rightarrow 0$ for some index i , we have

$$z_\mu \rightarrow \alpha_i \frac{y_{i\mu}}{y_i^2}, \quad z^2 \rightarrow \frac{\alpha_i^2}{y_i^2}, \quad \omega_N(x) \rightarrow \alpha_i \left(\frac{y_{i4} + i\vec{\sigma} \cdot \vec{y}_i}{y_i^2} \right) \frac{y_i}{\alpha_i} = \omega(y_i). \quad (63)$$

Now we transform the gauge field $A_\mu(x)$ using the function $\omega_N(x)$, this removes all the singularities at $x = a_i$. It follows

$$A'_\mu(x) = \omega_N(x)(A_\mu + \partial_\mu)\omega_N^{-1}(x). \quad (64)$$

As x approaches a_i , the gauge field A_μ becomes a pure gauge. In this limit,

$$A'_\mu(x) \rightarrow \omega(y_i)[\omega^{-1}(y_i)\partial_\mu\omega(y_i)]\omega^{-1}(y_i) + \omega(y_i)\partial_\mu\omega^{-1}(y_i) \quad (65)$$

$$= \partial_\mu[\omega(y_i)\omega^{-1}(y_i)], \quad (66)$$

where we used the pure gauge condition (60) and the $y_i \rightarrow 0$ limit for the function ω_N (63). The term related to A_μ in (64) vanishes and the singularities at $x = a_i$ are removed. Now we wish to prove that indeed this solution belongs to the N sector. After we removed the singularities of $A'_\mu(x)$, we can use the expression for the Pontryagin index (40):

$$N = -\frac{1}{16\pi^2} \int d^4x \text{Tr}(G'_{\mu\nu}\tilde{G}'_{\mu\nu})$$

where $G'_{\mu\nu}$ corresponds to the field tensor of $A'_\mu(x)$. Now that the integrand is non-singular, we may exclude from the integration region N small spheres with radii $\epsilon \rightarrow 0$ centered at $x = a_i$. For any small and finite ϵ , the integration is over Ω_ϵ , which consists of the entire space except the small spheres. The idea behind such procedure is that the original gauge field $A_\mu(x)$ is non-singular in the domain Ω_ϵ and thus $\text{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu}) = \text{Tr}(G'_{\mu\nu}\tilde{G}'_{\mu\nu})$, where $G_{\mu\nu}$ corresponds to the tensor field of A_μ . Therefore, we have

$$N = -\lim_{\epsilon \rightarrow 0} \frac{1}{16\pi^2} \int_{\Omega_\epsilon} d^4x \text{Tr}(G_{\mu\nu}\tilde{G}_{\mu\nu}),$$

this equation has the advantage of depending only on A_μ rather than A'_μ . Now that the singularities of A_μ have been removed from Ω_ϵ , we may use Gauss' theorem without ambiguities. The surfaces that bound the region Ω_ϵ are the sphere (S_∞) at $|x| \rightarrow \infty$ and the small surfaces σ_i of the N small spheres. Therefore, Gauss' theorem yields

$$N = \lim_{\epsilon \rightarrow 0} \left(\oint_{S_\infty} j_\mu d\sigma_\mu - \oint_{\sigma_1} j_\mu d\sigma_\mu - \oint_{\sigma_2} j_\mu d\sigma_\mu - \dots - \oint_{\sigma_N} j_\mu d\sigma_\mu \right)$$

where the current j_μ is defined, following Rajaraman (22), as:

$$j_\mu = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} \left[A_\nu \left(\partial_\alpha A_\beta + \frac{2}{3} A_\alpha A_\beta \right) \right], \quad (67)$$

the negative signs in equation (3.2) are associated with the convention that the internal surfaces point radially outward. As $|x| \rightarrow \infty$, the gauge field $A_\mu(x)$ goes as $1/|x|^3$ and thus j_μ goes to zero quickly enough that the contribution $\oint_{S_\infty} j_\mu d\sigma_\mu$ vanishes. On the other hand, when considering the small spheres, the gauge field $A_\mu(x)$ becomes a pure gauge with gauge function $\omega^{-1}(x)$ as $x \rightarrow a_i$ and the surface integral over σ_i gives the contribution:

$$N_i = \oint_{\sigma_i} j_\mu d\sigma_\mu = \frac{1}{24\pi^2} \int d\sigma_\mu \left(\frac{-12x_\mu}{|x|^4} \right) = -\frac{1}{2\pi^2} \int d\Sigma x_\mu |x|^2 \frac{x_\mu}{|x|^4} = -\frac{1}{2\pi^2} \int d\Sigma = -1 \quad (68)$$

where the solid angle element in the Euclidean space is $d\sigma_\mu = d\Sigma x_\mu |x|^2$. The contribution of any of the N small spheres is $\oint_{\sigma_i} j_\mu d\sigma_\mu = -1$, we can conclude our calculation of the limit (3.2), given that now we know the contribution of each term:

$$N = \lim_{\epsilon \rightarrow 0} [0 - (-1) - (-1) - \dots - (-1)] = n$$

since σ_i runs from $i = 1, \dots, n$. We obtained a solution, as it is self-dual, the action is given by $S = 8\pi^2 n/e^2$, exactly n times the action of a single instanton (56). It is remarkable that a non-linear equation (38) has solutions with a linear behavior.

4 Conclusion and final remarks

In this work, we explored some of the features of a $SU(2)$ Yang-Mills theory, deriving monopoles and instantons, which are classical solutions of physical interest. The monopole solutions arise when we consider the spontaneous symmetry breaking through the Higgs mechanism, where the configurations of fields at spatial infinity are mapped to the vacuum manifold by $\phi : S_\infty^2 \rightarrow \mathcal{M}_0$, those maps are characterized by the winding number, which is a topological invariant and is stable under homotopic transformations. In the process of symmetry breaking, the original gauge group $SU(2)$ is broken down to $U(1)$ and the gauge particles acquire mass proportional to the Higgs vacuum expectation value: $M = ae\hbar$. The connection between topological charge and the magnetic charge is made by identifying Maxwell's equations after the symmetry is spontaneously broken down. By using a radially symmetric ansatz, we calculated the equations of motion for the 't Hooft-Polyakov monopole, explored its asymptotic behavior, and showed that its magnetic charge is $g = 4\pi/e$, which satisfies Dirac's quantization condition $gg = 2\pi\hbar$. Thereupon, we derived the Bogomolny bound $M \geq a\sqrt{q^2 + g^2}$ and calculated the mass of the 't Hooft-Polyakov monopole, obtaining $M_g \geq 137M$, where M is the mass of the gauge particles. By considering the limit $|\vec{\phi}| \rightarrow a$ as $r \rightarrow \infty$ and $\lambda \rightarrow 0$ in the equations of motion for the radial ansatz, we derived the Bogomolny-Prasad-Sommerfield exact monopole solution and calculated its magnetic charge $g = 4\pi/e$ and mass $M = ag$. We briefly presented the ideas behind the Montonen-Olive conjecture. We considered a pure $SU(2)$ Yang-Mills theory on a 4-dimensional Euclidean space-time (\mathbb{E}^4) and showed that self-dual solutions minimize the action and thus satisfy the Yang-Mills equation. By using the ansatz of 't Hooft-Corrigan-Fairlie-Belavin we constructed the instanton solutions, following the argument of Giambiagi and Rothe, we generalized the ansatz for a N -instanton solution. We calculated the action of the instanton solution, obtaining $S = 8\pi/e^2$, which belongs to the $N = 1$ sector, and for the N -instanton solution we obtained $S = 8\pi n/e^2$, which corresponds to the $N = n$ sector and is n times the action of a single instanton. A remarkable property of the instanton solutions is their linear behavior, even though they come from the non-linear Yang-Mills equation.

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